We can **compute invariants** of moduli stacks using **geometric constructions**, corresponding to related **moduli problems**.

Computing motives of stacks of sheaves on stacky curves

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Sheaves on stacky curves

A stacky curve is a DM-stack that is a curve and generically a scheme. This simple definition has many interesting consequences.

- Stacky curves are iterated root stacks.
- A vector bundle on a stacky curve is equivalent to a (quasi-)parabolic bundle.
- A torsion sheaf is equivalent to a nilpotent quiver representation of a cyclic quiver.
- A coherent sheaf on a stacky curve has a *rank* and a *degree vector* related to both the parabolic flag type and the dimension vector of the quiver representation.

Main Theorem

The motive of the stack of coherent sheaves $\mathrm{Coh}_{n,\underline{d}}(\mathcal{C})$ of rank n and degree vector $\underline{d}$ on a stacky curve $\mathcal{C}$ is *pure* and generated by the motive of the curve.

Motives

There are many tools to compute motives.

- Let $V_0 \to V_1$ be vector bundles on X, then $M([V_1/V_0]) = M(X)$.
- Let $X \to Y$ be a cellular fibration with fiber F, then $M(X) = M(Y) \otimes M(F)$.
- Let $Z \hookrightarrow X$ be smooth closed open pair s.t. Z and U are pure, then $M(X) = M(Z)\{c\} \oplus M(U)$.
- Let $X \to Y$ be small, surjective and generically a G-torsor, then $M(Y) = M(X)^G$.

We can stratify by torsion type to reduce to torsion sheaves and vector bundles.

$$\mathrm{Coh}_{n,\underline{d}}(\mathcal{C}) = \coprod_{\underline{c}\geq 0} \mathrm{Coh}^{\mathrm{tor}=\underline{c}}_{n,\underline{d}}(\mathcal{C})$$

$$\mathrm{Coh}^{\mathrm{tor}=\underline{c}}_{n,\underline{d}}(\mathcal{C}) \to \mathrm{Coh}_{0,\underline{c}}(\mathcal{C}) \times \mathrm{Bun}_{n,\underline{d}-\underline{c}}(\mathcal{C})$$

The stack of vector bundles as a fibration.

$$\mathrm{Bun}_{n,\underline{d}}\left(\sqrt{p/C}\right) \xrightarrow{\sim} \mathrm{Flag}_{\underline{d}'}(p^*\mathcal{E}_{univ}) \downarrow \mathrm{Bun}_{n,d_0}(C)$$

A strange closed open pair.

$$[\mathbb{A}^{e-1}/\mathbb{G}_m^e] \hookrightarrow \mathrm{Coh}_{0,\underline{1}}\left(\sqrt[e]{p/C}\right) \hookleftarrow \mathrm{Coh}_{0,\underline{1}}\left(\sqrt[e-1]{p/C}\right)$$

The corresponding motivic formula.

$$M\left(\mathrm{Coh}_{0,\underline{1}}\left(\sqrt[e]{p/C}\right)\right) = M(B\mathbb{G}_m) \otimes \left(M(C) \oplus \bigoplus_{i=1}^{e-1} M(B\mathbb{G}_m^i)\{1\}\right)$$

Untwist the stack of torsion sheaves by considering the stack of filtered torsion sheaves.

$\underline{d} = a \cdot \underline{1} + \underline{b}$, with $\underline{b} \geq 0$ minimal.

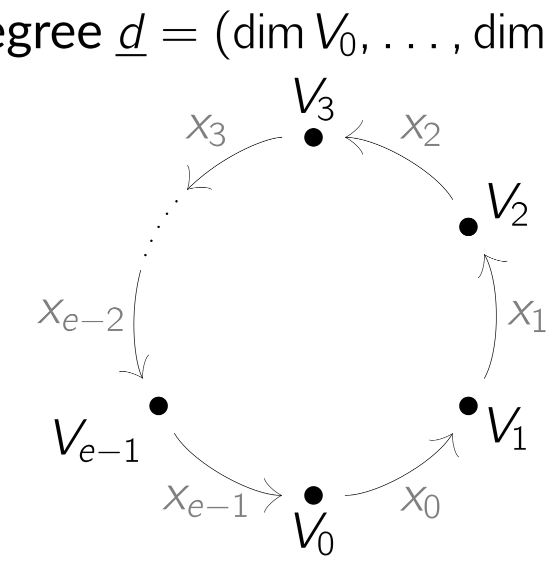
$$\widetilde{\mathrm{Coh}_{0,\underline{d}}}(\mathcal{C}) \longrightarrow \mathrm{Coh}_{0,\underline{b}}(\mathcal{C}) \times \prod_{i=1}^{a} \mathrm{Coh}_{0,\underline{1}}(\mathcal{C})$$

$$\widetilde{\mathrm{Coh}_{0,\underline{d}}}(\mathcal{C}) \xrightarrow{\approx S_a} \mathrm{Coh}_{0,\underline{d}}(\mathcal{C})$$

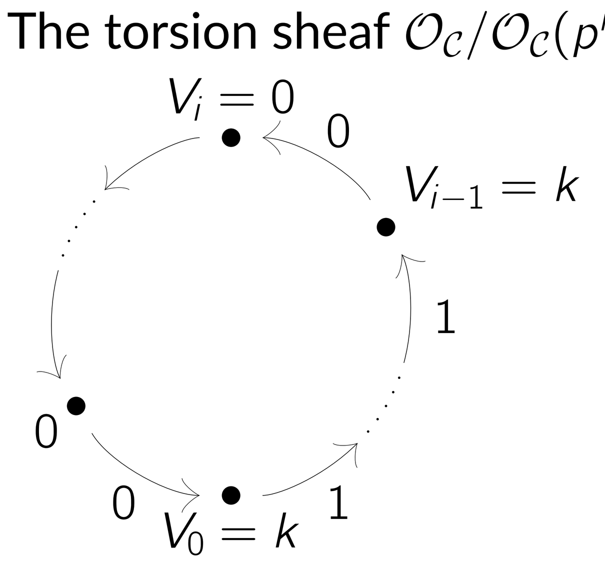
The corresponding motivic formula.

$$M\big(\mathrm{Coh}_{0,\underline{d}}(\mathcal{C})\big) = M\big(B\,\mathrm{GL}_{\underline{b}}\big) \otimes \mathrm{Sym}^a M\big(\mathrm{Coh}_{0,\underline{1}}(\mathcal{C})\big)$$

A torsion sheaf at a stacky point of order e. It has degree $\underline{d} = (\dim V_0, \ldots, \dim V_{e-1})$.



Example: The torsion sheaf $\mathcal{O}_{\mathcal{C}}/\mathcal{O}_{\mathcal{C}}(p^{i/e})$.



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